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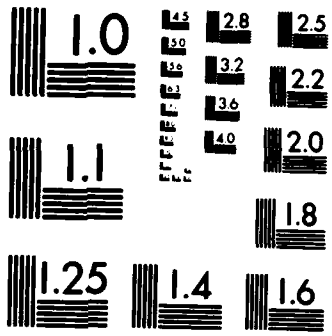

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A NOTE ON SHOT-NOISE AND RELIABILITY MODELING

by

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ABSTRACT

We discuss a reliability model which reflects the dynamic dependency between system failure and system stress. In particular, a shot-noise process is used to model "residual system stress," which in turn drives a doubly stochastic Poisson process model for system failures. Intuitively, residual stress (or susceptibility to failure) may vary in a random manner yet be essentially unobservable, while system failures may be readily detectable and observable. Shot-noise distributions have a richness and subtlety which suggest untapped potential for applications. The doubly stochastic Poisson process provides a reasonable framework for modeling randomly varying rates of occurrence in a broad variety of settings.

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3

1. INTRODUCTION

This note is motivated by the general approach to failure modeling discussed in Lemoine and Wenocur [4]. Under this approach, system state or wear and tear is modeled by an appropriately chosen random process; and the occurrence of fatal trauma is modeled by a Poisson process whose rate function is state dependent. The system is said to fail when either wear and tear accumulates beyond an acceptable or safe level or fatal trauma occurs. The rate at which fatal trauma can occur depends on the state of the system. The approach has interesting implications. It provides revealing insights into most of the famous and frequently used lifetime distributions in reliability theory, including the Makeham, Gompertz, Weibull, Rayleigh and Gumbel distributions; in fact, these classic models are obtained in a unified and straightforward manner. Moreover, the approach suggests intuitively appealing and computationally tractable ways of enhancing these standard failure models and for developing new ones.

In [4] detailed examples of the failure modeling approach are provided using deterministic models and diffusion process models for system state; use of a shot-noise model for system state is discussed in fairly general terms. In the present note we provide a more detailed discussion of the approach based on using a shot-noise model for system stress. And we assume that the rate of occurence of fatal trauma is proportional to "residual system stress". The time to system failure is then the epoch of the first count in a (doubly stochastic) Poisson process whose rate function is the shot-noise process.

In using a shot-noise model for system stress we assume that the system is subjected to "shots" or jolts according to a stochastic point process. A jolt may consist of an internal component malfunctioning or an external "blow" to the system. Jolts induce stress on the system when they occur. However, if the system survives the jolt it may recover to some extent. For instance, a sudden and unexpected surge of power in the circuit of a control system may temporarily increase the likelihood of system failure, but the overload itself decays rapidly. For another example, the mortality rate for persons who have suffered a heart attack declines with the elapsed time since the trauma. In this case, the heart actually repairs itself to a degree.

The remainder of this note is organized as follows. In the next section, Section 2, we derive a class of failure distributions using the shot-noise model for system stress, and discuss properties and structure. Following this, we discuss in Section 3 the shot-noise process and a related stochastic integral representation; this parallels the technical development in [4] based on stochastic differential equations and diffusion processes.

For background on shot-noise processes and distributions see Rice [5], Cox and Isham [2], and Bondesson [1]. Doubly stochastic Poisson processes are discussed by Grandell [3]. The gamma process is discussed in Takacs [6].

2. *SHOT-NOISE AND FAILURE DISTRIBUTIONS*

Suppose that a system is subjected to shots or jolts according to a Poisson process with deterministic rate function $\{m(u), u \geq 0\}$. Suppose that if a jolt of magnitude D occurs at epoch S then at time $S+t$ the contribution of the jolt to the system stress is $Dh(t)$, where h is a non-negative function which tends to zero as $t \to \infty$. (The function h is also assumed to vanish on $(-\infty, 0)$ and to be integrable over finite intervals.) In other words, shot-induced stress is additive and decays with time according to the rate function h. If $\{S_n, n \geq 1\}$ are the epochs of shot occurrences and $\{D_n, n \geq 1\}$ are the magnitudes of the successive jolts, then the "residual system stress" at time t, say $X(t)$, is given by

$$X(t) = \sum_{n=1}^{\infty} D_n h(t-S_n) \tag{1}$$

where we take $X(0)=0$. With stress comes susceptibility to failure, and it is assumed that the occurrence rate of fatal trauma is an increasing function of residual system stress. In particular, if T is the time to system failure, we assume that $P\{ T \leq \delta + u \mid T > u, X(u) = x \} = \delta k x + o(\delta)$ for each time point u and some parameter k, so that kx can be interpreted as the rate of occurrence of fatal trauma when the level of system stress is x. Thus, the failure time T is the epoch of the first count in a Poisson process with rate function $\{kX(u), u \geq 0\}$, and therefore

$$P\{ T > t \} = E[\exp[-\int_0^t kX(u)du]]. \tag{2}$$

The usual choice for the attenuation or recovery function h in the literature is the function $h(u)=\exp(-au)$ if $u \geq 0$ and $h(u) = 0$ if $u < 0$, where a is some positive parameter, that is, exponential decay. Alternate choices for modeling the pattern of shot occurrences include a renewal process, a semi-Markov process, or a cluster point process. In this note we assume a Poisson pattern for shot occurrences. We also take the shot occurrences and jolt magnitudes to be independent of one another, and the successive jolt magnitudes to be independent and distributed as a random variable D which is not identically 0 and has a proper distribution.

We now compute the right side of equation (2) and related quantities, discuss properties of the resulting formulas, and then look at some examples.

To begin, let $N(t)$ be the number of shot occurrences up to time t,

$$M(t) = \int_0^t m(u)du \tag{3}$$

and

$$Y(t) = \int_0^t kX(u)du. \tag{4}$$

We assume that $E\{N(t)\} = M(t)$ is finite for each t, from which it follows that $Y(t)$ is finite with probability one. Put

$$H(t) = \int_0^t h(u)du. \tag{5}$$

If $N(t) = n$, then

$$Y(t) = \sum_{j=1}^{n} kD_j H(t-S_j) \tag{6}$$

and the shot epochs $S_1, \cdots, S_n$ are independent and distributed as a random variable U with density $m(u)/M(t)$ for $0 \leq u \leq t$. Thus, if L is the Laplace transform of the distribution of D, then for $s \geq 0$

$$\begin{aligned} E\{\exp[-sY(t)] \,|\, N(t)=n\} &= \left(E\{\exp[-skDH(t-U)]\}\right)^n \\ &= \left(\int_0^t L(skH(t-U))[m(u)/M(t)]du\right)^n. \end{aligned} \tag{7}$$

Unconditioning on the value of $N(t)$, we find that

$$E\{\exp[-sY(t)]\} = \exp[-M(t)]\exp[\int_0^t L(skH(u))m(t-u)du]. \tag{8}$$

Putting $s=1$ in equation (8) gives

$$P\{T>t\} = \exp[-M(t)]\exp[\int_0^t L(kH(u))m(t-u)du]. \tag{9}$$

If $M(t) \to \infty$ as $t \to \infty$ then equation (9) defines a proper probability distribution concentrated on $(0,\infty)$. We assume henceforth that this is the case. Let $R(t)$ denote the quantity on the right side of equation (9). If the function $m(\cdot)$ driving the process of shot occurrences is almost everywhere differentiable, then the distribution of T has density $R(t)r(t)$ and failure rate $r(t)$, where

$$r(t) = m(t) - m(0)L(kH(t)) - \int_0^t L(kH(u))m'(t-u)du. \tag{10}$$

Now consider $N(T)$, the number of jolts leading to system failure. We indicate briefly how to obtain the generating function of $N(T)$. We begin with the joint distribution of T and $N(T)$. Consider the difference

$$P\{t-h < T \leq t\,,\, N(t)=n\} = P\{T>t-h, N(t)=n\} - P\{T>t, N(t)=n\} \qquad (11)$$

Formally, if we divide equation (11) by h and let h tend to 0, we obtain the joint density of T and $N(T)$. This limiting process can be rigorously justified by appealing to the absolute continuity of the distribution of T. We now proceed by expanding the left-hand-side of (11) into more tractable quantities. Standard probabilistic arguments lead to:

$$P\{T>t-h\,,\, N(t)=n\} - P\{T>t, N(t)=n\} = g(t-h,n) - g(t,n) \qquad (12)$$

$$-g(t-h,n)[1-P\{N(t)-N(t-h)=0\}] + g(t-h,n-1)P\{N(t)-N(t-h)=1\} + o(h),$$

where $g(t,n) = P\{T>t\,,\, N(t)=n\}$. Dividing equation (12) through by h and letting $h \to 0$ gives

$$P(T \in dt\,,\, N(T)=n) = [g(t,n-1)-g(t,n)]m(t)dt - g'(t,n)dt. \qquad (13)$$

Now

$$g(t,n) = [\int_0^t L(kH(u))m(t-u)du]^n \exp[-M(t)]/n!\ , \qquad (14)$$

and substituting (14) into equation (13) yields

$$P(T \in dt\,,\, N(T)=n) = g(t,n-1)r(t)dt\,. \qquad (15)$$

A simple integration and summation now show that for $0 \leq \theta \leq 1$,

$$E[\theta^{N(T)}] = \int_0^\infty \theta \exp[\theta \int_0^t L(kH(u))m(t-u)du] \exp[-M(t)]r(t)dt. \qquad (16)$$

Finally, differentiating (16) with respect to θ and setting $\theta = 1$ gives

$$E[N(T)] = 1 + \int_0^\infty \left(\int_0^t L(kH(u))m(t-u)du \right) R(t)r(t)dt. \qquad (17)$$

Other quantities of interest are the mean and variance of $Y(t)$. Differentiating the tranform (8) with respect to s and then setting $s = 0$, we see that

$$E\{Y(t)\} = E\{D\} \int_0^t kH(u)m(t-u)du, \tag{18}$$

and if $E\{Y(t)\}$ is finite,

$$Var\{Y(t)\} = E\{D^2\} \int_0^t [kH(u)]^2 m(t-u)du. \tag{19}$$

The most tractable circumstance occurs when $m(\cdot)$ is a step function or constant, in which case equation (10) reduces to

$$r(t) = m(t) - m(0)L(kH(t)). \tag{20}$$

Since this case covers a multitude of possibilities, we assume henceforth that the rate function driving the shot process is either a step function or constant. To preclude tedious nuisance, we assume that $m(0) = m(0+) > 0$. We now consider some specific examples.

Example 1: Suppose that the jolt magnitudes are constant and that the jolt-induced stress decays slowly. In particular, suppose that $D \equiv d$ and that $h(u)=1/(1+u)$ for $u \geq 0$. Then

$$r(t) = m(t) - [m(0)/(1+t)^{kd}], \tag{21}$$

$$R(t) = \exp[-M(t)](1+t)^{m(0)} \tag{22}$$

if $kd = 1$, and

$$R(t) = \exp[-M(t)]\exp[m(0)\{(1+t)^{1-kd} - 1\}/(1-kd)] \tag{23}$$

otherwise. If $M(t)=mt$, then (16) reduces to

$$R(t) = \exp[-mt](1+t)^m. \tag{24}$$

Example 2: Suppose that D has a geometric distribution; specifically, suppose that $P\{D = n\} = qp^n$ for $n = 0,1,2,\ldots$, where $0 < p < 1$. Assume that stress decays slowly as in Example 1, and that $k = 1$. Then

$$r(t) = m(t) - m(0)q[(1 + t)/(q + t)] \qquad (25)$$

and

$$R(t) = \exp[m(0)qt - M(t)] \{(q + t)/q\}^{m(0)qp}. \qquad (26)$$

Example 3: Suppose there is a delayed reaction to a jolt, followed by slow decay; in particular, assume that $h(v) = \ln(1 + av)/(1 + av)$ for $v \geq 0$, where a is some positive parameter. Further, suppose that the distribution of D has a heavy tail; in particular, assume the distribution of D has transform $\exp[-b(2s)^{1/2}]$ for $s \geq 0$ where b is some positive parameter. Putting $c = ba^{-1/2}$ and $k = 1$ gives

$$r(t) = m(t) - m(0)\,[1/(1 + at)^c], \qquad (27)$$

which is very similar to (21). If also $M(t) = mt$ and $c = 1$ then

$$R(t) = \exp[-mt](1 + at)^m. \qquad (28)$$

Example 4: Suppose that shot-induced stress decays at an exponential rate; in particular, assume $h(v) = \exp[-av]$ for $v \geq 0$, where a is some positive parameter. Further, suppose that D has an exponential distribution with parameter b. Then

$$r(t) = m(t) - m(0)\,\{ab/(ab + k - k\exp[-at])\} \qquad (29)$$

and

$$R(t) = \exp[\{m(0)abt/(ab + k)\} - M(t)] \cdot$$

$$\{(ab + k - k\exp[-at]) / ab\}^{m(0)b/(ab+k)} \qquad (30)$$

In particular, if $M(t) = mt$ and $k = 1$, then

$$R(t) = \exp[-mabt/(1 + ab)]\,\{(1 + ab - \exp[-at])/ab\}^{mb/(1 + ab)}. \qquad (31)$$

In this last case, residual system stress $X(t)$ has a limit distribution (as $t \to \infty$) which is gamma with location parameter b and shape parameter m/a; cf. Bondesson [1]. This is a remarkable result in that a gamma distribution with arbitrary shape parameter arises in a physical model.

3. A GAMMA PROCESS DRIVING SYSTEM STRESS

Let $\{Z(u), u \geq 0\}$ be a compound Poisson process with jump-rate function $\{m(u), u \geq 0\}$ and jump sizes distributed as D. Note that the series representation for $X(t)$ given by (1) truncates at $N(t)$, the number of jolts up to time t. Thus, $X(t)$ admits the equivalent representation

$$X(t) = \int_0^t h(t-u)Z(du) \tag{32}$$

where the right side of (32) is a Riemann-Stieltjes integral. This suggests replacing the compound Poisson process $Z(\cdot)$ to drive system stress by some other process which also has independent and positive increments. In particular let $\{Z^*(u), u \geq 0\}$ be a gamma process (cf. [6]) for which

$$E\{\exp[-sZ^*(u)]\} = \{b/(b+s)\}^u \tag{33}$$

for all $u, s \geq 0$, where b is some positive parameter. (The increments of $Z^*(\cdot)$ are in fact stationary.) Now let

$$X^*(t) = \int_0^t h(t-u)Z^*(du) \tag{34}$$

and assume that the residual system stress at time t is expressed by the Riemann-Stieltjes integral on the right side of (34). Further, suppose that the system failure time T^* is the epoch of the first count in a Poisson process with rate function $\{kX^*(u), u \geq 0\}$. If $R^*(t) = P\{T^* > t\}$ then

$$R^*(t) = E\{\exp[-\int_0^t kX^*(u)du]\} \tag{35}$$

To compute the right side of (35), we first note that

$$\int_0^t X^*(u)du = \int_0^t H(t-u)Z^*(du), \tag{36}$$

where $H(\cdot)$ is given by (5). If J is a positive integer then

$$\ln(E\{\exp\{-k\sum_{j=1}^{J} H(t-(jt/J))[Z^*(jt/J)-Z^*((j-1)t/J)]\}\}) = \sum_{j=1}^{J}(t/J)\ln\{b/[b+kH(t-(jt/J))]\}. \tag{37}$$

Letting $J \to \infty$ on both sides of (37) yields

$$R^*(t) = \exp\left[\int_0^t \ln\{b/(b+kH(u))\}du\right]. \tag{38}$$

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